JERK LIMITED VELOCITY PROFILE GENERATION FOR HIGH SPEED INDUSTRIAL ROBOT TRAJECTORIES

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Abstract: Trapezoidal velocity profiles are no longer sufficient for recent high speed industrial robots involved in precision works. Most of the recently developed trajectory generation algorithms have adopted jerk limited profiles, where computation of the coefficients is intricate. The previous algorithms have attempted either to obtain suboptimal solutions or to solve the problem in an iterative manner which complicates on-line implementation. This paper presents analytic solutions to the computational problem. The solutions presented are derived from the simultaneous equations for displacement condition and peak velocity condition, where jerk, acceleration and velocity constraints are resolved. By including only the low-order polynomial equations that can be evaluated analytically, the proposed algorithm is readily implementable on the fly.

Keywords: robot, trajectory, jerk, velocity profile

1. INTRODUCTION

Modern industrial robots operating in high-speed with precision require very smooth trajectory generation because small discontinuities in the reference trajectory may result in undesirable high frequency harmonics, which end up in exciting the natural modes of the robotic system. However, a typical joint system of industrial robots actuated with electrical AC servo motors has several physical limits. Jerk is limited because the current in the motor cannot be changed instantly. Acceleration and deceleration is limited because of the inertia of the mechanical system as well as the controller dynamics. Furthermore, velocity is also limited by kinetic friction or has to be reduced for safe operation.

To solve these problems, a significant amount of effort has been devoted to developing new algorithms that provide smooth trajectories. Butler et al. (Butler *et al.*, 1988) have recommended the modification of the feedrate profile for a given path of second order continuity to avoid actuator saturations. Weck and Ye (Weck and Ye, 1990) introduced low-pass to remove the high frequency components of reference trajectories. Wang and Yang (Wang and Yang, 1993) have implemented trajectory generation via cubic and quintic splines. Simon and Isik (Simon and Isik, 1991) have proposed an algorithm using trigonometric splines.

Recently, Erkorkmaz and Altintas (Erkorkmaz and Altintas, 2001) have proposed a noticeable algorithm using a fifth order resampling technique with a jerk limited speed control, where the proposed structure is ready for generation of jerk limited time-optimal trajectory along the specified tool path. Although noticeable, their algorithm involves iterative steps in deciding the coefficients of jerk limited speed profile. Besides, their coefficient decision algorithm is limited to use of speed profile only.

This paper presents an algorithm that can determine the coefficients of jerk limited profiles without these defects. Since the proposed algorithm does not have non-negative velocity and displacement constraints, it can be used to generate velocity profiles for point-to-point motion trajectories as well as speed profiles for path motion trajectories. For example, time-optimal point-to-point trajectory can be generated with the following steps.

Step 1. Generate jerk limited time-optimal trajectory for each joint.
Step 2. Determine the optimal traveling period T^* free from quantization errors using (1), where t_L denotes the longest traveling period among the joints and Δt denotes sampling time.

$$T^* = \Delta t \left\lceil \frac{t_L}{\Delta t} \right\rceil \tag{1}$$

Step 3. Generate jerk limited time-fixed trajectory for each joint with traveling period constraint T^*.

The proposed algorithm offers analytic solutions to determine the coefficients of jerk limited velocity profiles for time-optimal cases as well as time-fixed cases. Consequently, as trajectories are generated analytically rather than iteratively, the proposed algorithm is readily implementable online.

The rest of this paper is organized as follows: Problem formulation is explained in Section 2. Section 3 presents solutions to jerk limited time-optimal velocity profile problem, and Section 4 presents solutions to time-fixed problem. The Conclusions are summarized in Section 5.

2. PROBLEM FORMULATIONS

Since the paper considers *jerk limited velocity* profiles only, the modifier *jerk limited velocity* for profiles will be omitted except for special purposes. It is notable how the velocity profile generation problem is related with the physical joint systems. The problems are obtained by investigating joint systems from the two points of view: dynamics constraints and boundary conditions. The dynamics constraints are the limits imposed on the maximum jerk, acceleration, deceleration, and velocity by the physical properties of the joint actuators and robot dynamics. The boundary conditions are the initial and final values of positions and velocities. For second order continuity of a profile, the boundary conditions for acceleration are required, but they are excluded because the boundary conditions for position and velocity are sufficient when field applications are considered. Since it is almost impossible to measure the exact accelerations, and especially since boundary conditions of accelerations result in a very complex algorithm for profile generation, the boundary conditions of accelerations will be considered as zero. Viewed from these two different aspects, we have the following problems.

Problem 2.1. (time-optimal profile). For a set of given constraints $\{p_0,\ p_f,\ v_0,\ v_f,\ V,\ A,\ D,\ J\}$, find the profile $p(t)$ that minimizes the traveling period T while satisfying the following boundary conditions and limit constraints (2).

$$\begin{cases} p(0) = p_0, p(T) = p_f, \\ \dot{p}(0) = v_0, \dot{p}(T) = v_f, \\ \ddot{p}(0) = 0, \ddot{p}(T) = 0 \end{cases}, \begin{cases} -V \le \dot{p}(t) \le V \\ -D \le \ddot{p}(t) \le A \\ -J \le p^{(3)}(t) \le J \end{cases} \tag{2}$$

Problem 2.2. (time-fixed profile). For a set of given constraints $\{p_0,\ p_f,\ v_0,\ v_f,\ V,\ A,\ D,\ J,\ T\}$, find a profile $p(t)$ that satisfies the boundary conditions and the limit constraints (2), provided that T is longer than the traveling period of the jerk limited time-optimal profile under the same constraints excluding T.

We follow the Erkorkmaz and Altintas's formulations in (Erkorkmaz and Altintas, 2001) for profile description. In (Erkorkmaz and Altintas, 2001), they generated jerk limited feedrate profiles for smooth trajectories of CNC machine tools. Although their algorithm for feedrate profiles can not be used for the velocity profiles of joint trajectories because the feedrate and travel length of CNC machines are always positive while both of them can be negative in joint trajectories, the formulations are still valid for the velocity profiles.

Erkorkmaz and Altintas formulated jerk limited profiles as piecewise polynomials. Figure 1 shows a typical example of jerk limited profile. Equation (3) is the formulation of acceleration, velocity, and position profile, denoted as $\ddot{p}(t), \dot{p}(t)$, and $p(t)$, respectively. In the equation, each time point t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7 denotes the time point of the beginning acceleration region, the ending of the constant acceleration region, the beginning of the constant velocity, the ending of the constant velocity region, the beginning of the constant deceleration region, the ending of the constant deceleration, and the ending of profile, where, without loss of generality, t_0 is assumed to be zero. p_k and v_k denote the position and the velocity at each time point, while τ_k and T_k are defined as $\tau_k = t - t_{k-1}, T_k = t_k - t_{k-1}$.

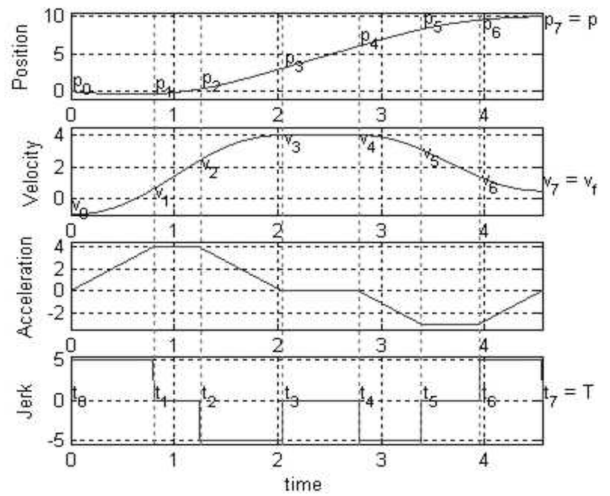

Fig. 1. A typical example of jerk-limited profile

The profile generation problems are to decide the time intervals of each region T_k or each time point t_k while adjusting the actual limits of jerk, acceleration, and deceleration.

$$\ddot{p}(t) = \begin{cases} J\tau_1 \\ A \\ A - J\tau_3 \\ 0 \\ -J\tau_5 \\ -D \\ -D + J\tau_7 \end{cases}, \dot{p}(t) = \begin{cases} v_0 + 0.5J\tau_1^2 \\ v_1 + A\tau_2 \\ v_2 + A\tau_3 - 0.5J\tau_3^2 \\ v_3 \\ v_4 - 0.5J\tau_5^2 \\ v_5 - D\tau_6 \\ v_6 - D\tau_7 + 0.5J\tau_7^2 \end{cases}$$
$$p(t) = \begin{cases} p_0 + v_0\tau_1 + J\tau_1^3/6, & t_0 \le t < t_1 \\ p_1 + v_1\tau_2 + 0.5A\tau_2^2, & t_1 \le t < t_2 \\ p_2 + v_2\tau_3 + 0.5A\tau_3^2 - J\tau_3^3/6, & t_2 \le t < t_3 \\ p_3 + v_3\tau_3, & t_3 \le t < t_4 \\ p_4 + v_4\tau_5 - J\tau_5^3/6, & t_4 \le t < t_5 \\ p_5 + v_5\tau_6 - 0.5D\tau_6^2, & t_5 \le t < t_6 \\ p_6 + v_6\tau_7 - 0.5D\tau_7^2 + J\tau_7^3/6, & t_6 \le t < t_7 \end{cases} \quad (3)$$

In the above equations, p_k and v_k are defined as follows.

$$\begin{cases} p_1 = p_0 + v_0T_1 + JT_1^3/6 \\ p_2 = p_1 + v_1T_2 + 0.5AT_2^2 \\ p_3 = p_2 + v_2T_3 \\ \qquad +0.5AT_3^2 - JT_3^3/6 \\ p_4 = p_3 + v_3T_4 \\ p_5 = p_4 + v_4T_5 - JT_5^3/6 \\ p_6 = p_5 + v_5T_6 - 0.5DT_6^2 \end{cases}, \begin{cases} v_1 = v_0 + 0.5JT_1^2 \\ v_2 = v_1 + AT_2 \\ v_3 = v_2 + AT_3 - 0.5JT_3^2 \\ v_4 = v_3 \\ v_5 = v_4 - 0.5JT_5^2 \\ v_6 = v_5 - DT_5 \end{cases}$$

We introduce the terminology for simpler formulation. Profiles are classified into two groups on the basis of whether or not the acceleration regions precede the deceleration regions. Profiles whose acceleration regions precede the deceleration regions are denoted as AFP(acceleration-first-profile), and the others as DFP(deceleration-first-profile). The following fact is useful in deciding if the resultant profile is AFP or DFP under a given constraint set.

Fact 1. (Properties of AFP and DFP). An AFP $P(t)$ under the constraint set $\{p_0,\ p_f,\ v_0,\ v_f,\ V,\ A,\ D,\ J\}$ satisfies the followings.

1. Average velocity is greater than the mean value of the initial and final velocities.
$$(p_f - p_0)/T \ge (v_0 + v_f)/2 \quad (4)$$
2. If $Q(t)$ is a DFP whose constraint set is $\{-p_0,\ -p_f,\ -v_0,\ -v_f,\ V,\ D,\ A,\ J\}$, $Q(t)$ has the following relations with $P(t)$.
$$Q(t) = -P(t), \dot{Q}(t) = -\dot{P}(t), \ddot{Q}(t) = -\ddot{P}(t) \quad (5)$$
3. If $p'_f - p'_0 \ge p_f - p_0$,there is an AFP solution under a new constraint set $\{p'_0,\ p'_f,\ v_0,\ v_f,\ V,\ A,\ D,\ J\}$.
4. If there is not an AFP solution under a given constraint set, there is a DFP solution under the same constraint set.

Eventually, we will solve DFP problems after converting them to AFP problems using this.

3. TIME OPTIMAL SOLUTION

This section develops analytic solutions to Problem 2.1. The solutions are developed for AFP problems only since AFP problems and DFP problems can be transformed into each other. The method of deciding if the given problem is AFP or not is explained in Subsection 3.1.

We divide the time intervals of an AFP into three regions: acceleration region, constant velocity region, and deceleration region, where the period of each region is denoted as $x = T_1 + T_2 + T_3, \hat{x} = T_4, \bar{x} = T_5 + T_6 + T_7$. Because the acceleration profile of each region is symmetric, we can determine all the time points $t_1, \cdots, t_7$ using the periods $x, \hat{x}, \bar{x}$ and the peak velocity, denoted as v_p, at the constant velocity region.

In order to find $x, \hat{x}, \bar{x}$, we establish the simultaneous equations of displacement condition and peak velocity condition. The displacement condition is given as follows:

$$L = 0.5(v_0 + v_p)x + 0.5(v_p + v_f)\bar{x} + v_p\hat{x} \quad (6)$$

The peak velocity condition is so complicated that it will be explained later via divide-and-conquer approach.

When the proper values of $x, \hat{x}, \bar{x}$, and v_p are acquired, they enable us to determine the coefficient of profile following the steps below.

First, readjust the acceleration and deceleration limits using (7), if the constant acceleration or deceleration region does not exist.

$$\begin{matrix} A \leftarrow 0.5Jx, \text{ if } v_p - v_0 > 0.25Jx^2 (\text{no const. acc.}) \\ D \leftarrow 0.5J\bar{x}, \text{ if } v_p - v_f > 0.25J\bar{x}^2 (\text{no const. dec.}) \end{matrix} \quad (7)$$

Next, with the adjusted limits, determine the periods $T_1, \cdots, T_7$ using (8).

$$T_1 = A/J, T_2 = x - 2T_1, T_3 = T_1, T_4 = \hat{x}, \\ T_5 = D/J, T_6 = \bar{x} - 2T_5, T_7 = T_5 \tag{8}$$

3.1 Decision if AFP or DFP

In the case of a DFP problem, we need to transform it into an AFP problem after deciding AFP or DFP. The transformation can be performed using Fact 1, where the third and fourth properties of Fact 1 are used for decision since T is not known at the point of deciding. The decision procedure is explained below.

First, we need to find the minimum time T_m to change velocity from v_0 to v_f under the given constraints A, D, and J, where the consequent profile is denoted as $P_m(t)$. $P_m(t)$ is either an acceleration only profile or a deceleration only profile. Depending on whether $P_m(t)$ is acceleration profile or deceleration one, and depending on whether a constant acceleration or deceleration region exists in it, T_m is computed in four cases.

$$T_m = \begin{cases} 2\sqrt{(v_f - v_0)/J}, & \text{if } v_0 \le v_f, v_f - v_0 \le \frac{A^2}{J} \\ (v_f - v_0)/A + A/J, & \text{if } v_0 \le v_f, v_f - v_0 > \frac{A^2}{J} \\ 2\sqrt{(v_0 - v_f)/J}, & \text{if } v_0 > v_f, v_0 - v_f \le \frac{D^2}{J} \\ (v_0 - v_f)/D + D/J, & \text{if } v_0 > v_f, v_0 - v_f > \frac{D^2}{J} \end{cases} \tag{9}$$

Then, we can decide if the time-optimal profile is AFP or not by evaluating the following conditions, where $L_m = \frac{v_0+v_f}{2} T_m$ means displacement of $P_m(t)$ during $0 \le t \le T_m$.

$$\text{AFP} : (v_0 \le v_f, L_m \le L) \text{ or } (v_0 > v_f, L_m < L) \\ \text{DFP} : (v_0 \le v_f, L_m > L) \text{ or } (v_0 > v_f, L_m \ge L) \tag{10}$$

3.2 Evaluation of periods

Equations for $x, \hat{x}, \bar{x}$ are presented in this subsection using the simulations equations of the displacement and the peak velocity condition. The peak velocity condition is roughly divided into two cases according to whether or not the constant velocity region exists. Each case is again subdivided into four cases according to the existence of constant acceleration region T_2 or deceleration region T_6. Since we cannot assert to which case the time-optimal profile belongs at this point, we need to evaluate equations and validate the corresponding assumptions one by one until the valid solution is found.

Case of no constant velocity region

The peak velocity condition and the resultant solution of simultaneous equations are arranged together with validation inequalities for $x, \bar{x}$, which come from the assumptions of each case.

CASE ($T_4 = 0, T_2 > 0, T_6 > 0$)

$$v_p = v_0 - \frac{A^2}{J} + Ax = v_f - \frac{D^2}{J} + D\bar{x} \tag{11}$$

$$A(\frac{A}{D} + 1)x^2 + \frac{1}{JD}(A + D)(AD - 2A^2 + 2v_0 J)x \\ -2L - \frac{1}{D}(v_0 + v_f - \frac{A^2}{J})(v_f - v_0 + \frac{A^2 - D^2}{J}) = 0 \tag{12}$$

$$x \ge \frac{2A}{J}, \bar{x} \ge \frac{2D}{J} \tag{13}$$

CASE ($T_4 = 0, T_2 = 0, T_6 > 0$)

$$v_p = v_0 + \frac{1}{4}Jx^2 = v_f - \frac{D^2}{J} + D\bar{x} \tag{14}$$

$$\frac{J^2}{16D}x^4 + \frac{1}{4}Jx^3 + \frac{1}{4}(2\frac{Jv_0}{D} + D)x^2 + \\ 2v_0 x - 2L + \frac{1}{D}(v_0 + v_f)(v_0 - v_f + \frac{D^2}{J}) = 0 \tag{15}$$

$$0 \le x < \frac{2A}{J}, \bar{x} \ge \frac{2D}{J} \tag{16}$$

CASE ($T_4 = 0, T_2 > 0, T_6 = 0$)

$$v_p = v_0 - \frac{A^2}{J} + Ax = v_f + \frac{1}{4}J\bar{x}^2 \tag{17}$$

$$\frac{J^2}{16A}\bar{x}^4 + \frac{1}{4}J\bar{x}^3 + \frac{1}{4}(2\frac{Jv_f}{A} + A)\bar{x}^2 + \\ 2v_f\bar{x} - 2L + \frac{1}{A}(v_f + v_0)(v_f - v_0 + \frac{A^2}{J}) = 0 \tag{18}$$

$$x \ge \frac{2A}{J}, 0 \le \bar{x} < \frac{2D}{J} \tag{19}$$

CASE ($T_4 = 0, T_2 = 0, T_6 = 0$)

$$v_p = v_0 + \frac{1}{4}Jx^2 = v_f + \frac{1}{4}J\bar{x}^2 \tag{20}$$

$$\frac{1}{4}(v_f - v_0)Jx^4 + JLx^3 - (v_f - v_0)^2x^2 \\ +8v_0 Lx - 4\{L^2 + \frac{1}{J}(v_0 + v_f)^2(v_f - v_0)\} = 0 \tag{21}$$

$$0 \le x < \frac{2A}{J}, 0 \le \bar{x} < \frac{2D}{J} \tag{22}$$

Cases of constant velocity region

In the case of a constant velocity region, the peak velocity v_p is V and $\hat{x} \ge 0$. The validation inequalities are the same as those for the cases of no constant velocity region except for the additional equation of (23), hence, omitted.

$$\hat{x} = \frac{2L - (v_0 + V)x - (V + v_f)\bar{x}}{2V} \ge 0 \tag{23}$$

CASE ($T_4 > 0, T_2 > 0, T_6 > 0$)

$$V = v_0 - \frac{A^2}{J} + Ax = v_f - \frac{D^2}{J} + D\bar{x} \tag{24}$$

$$x = \frac{V - v_0}{A} + \frac{A}{J}, \ \bar{x} = \frac{V - v_f}{D} + \frac{D}{J} \tag{25}$$

CASE ($T_4 > 0, T_2 = 0, T_6 > 0$)

$$V = v_0 + \frac{1}{4}Jx^2 = v_f - \frac{D^2}{J} + D\bar{x} \tag{26}$$

$$x = 2\sqrt{\frac{V - v_0}{J}}, \ \bar{x} = \frac{V - v_f}{D} + \frac{D}{J} \tag{27}$$

CASE ($T_4 > 0, T_2 > 0, T_6 = 0$)

$$V = v_0 - \frac{A^2}{J} + Ax = v_f + \frac{1}{4}J\bar{x}^2 \tag{28}$$

$$x = \frac{V - v_0}{A} + \frac{A}{J},\ \bar{x} = 2\sqrt{\frac{V - v_f}{J}} \tag{29}$$

CASE ($T_4 > 0, T_2 = 0, T_6 = 0$)

$$V = v_0 + \frac{1}{4}Jx^2 = v_f + \frac{1}{4}J\bar{x}^2 \tag{30}$$

$$x = 2\sqrt{\frac{V - v_0}{J}},\ \bar{x} = 2\sqrt{\frac{V - v_f}{J}} \tag{31}$$

4. TIME-FIXED SOLUTION

While a time-optimal problem always has a single solution that minimizes the traveling period T, any profile that satisfies the boundary conditions (p_0, p_f, v_0, v_f) and the limit constraints (V, A, D, J) becomes a solution of the problem as long as its traveling period is T. In other words, additional criteria are required for choosing a solution. The jerk-minimizing criterion is proposed in this paper, because, as jerk becomes smaller, acceleration profile becomes smoother. Symbol γ is introduced to demonstrate the relation between a given jerk limit J and the effective jerk limit J_e, where $J_e = \gamma J$.

The procedure to find a time-fixed solution is similar to that of time-optimal solution. First, determine whether the resultant profile is AFP or DFP. Since the traveling period T is given, one can simply decide it using (4). If the profile is DFP, transform the problem into the corresponding AFP problem. Second, by evaluating the simultaneous equations of the displacement condition and the peak velocity condition, find $x, \bar{x}$ and γ. The detailed procedure is described later. In any case, $x, \bar{x}$ and γ should satisfy (6), (32a), and (32b).

$$0 \le x, \bar{x}, T - x - \bar{x} \le T, \tag{32a}$$

$$0 < \gamma \le 1 \tag{32b}$$

Third, readjust the acceleration, deceleration, and jerk limits using equations (33a)∼(33c), if needed.

$$A \leftarrow 0.5Jx,\ \text{if } x < \frac{2A}{\gamma J} \text{ (no const. acc.)} \tag{33a}$$

$$D \leftarrow 0.5J\bar{x},\ \text{if } \bar{x} < \frac{2D}{\gamma J} \text{ (no const. dec.)} \tag{33b}$$

$$J = J_e \leftarrow \gamma J \tag{33c}$$

Finally, determine each time interval using (8) with the adjusted A, D, J.

Cases of no constant velocity region

With the similar approaches used for time-optimal solutions, we divide the peak velocity condition into four cases. For each case, we arrange the peak velocity condition, the equation for x or $\hat{x}$, and the equation for γ. The validating equations are omitted since they are very similar to those in the time-optimal solutions, and since they can be obtained by replacing J of those with γJ.

CASE ($T_4 = 0, T_2 > 0, T_6 > 0$)

$$v_p = v_0 - \frac{A^2}{\gamma J} + Ax = v_f - \frac{D^2}{\gamma J} + D\bar{x} \tag{34}$$

$$x = \begin{cases} \dfrac{\left(\frac{A^2(v_0 - v_f - DT)}{D^2 - A^2} + v_0 + v_f\right)T - 2L}{\left(\frac{A^2}{D - A} + A\right)T + v_0 - v_f} & , A \ne D \\ \dfrac{1}{2A}(v_f - v_0 + AT) & , A = D \end{cases} \tag{35}$$

$$\gamma = \begin{cases} -\dfrac{D^2 - A^2}{J\{(A + D)x + v_0 - v_f - DT\}} & , A \ne D \\ \dfrac{A^2 T}{J\{(v_0 + v_f)T + (v_0 - v_f + AT)x - 2L\}} & , A = D \end{cases} \tag{36}$$

CASE ($T_4 = 0, T_2 = 0, T_6 > 0$)

$$v_p = v_0 + \frac{1}{4}\gamma Jx^2 = v_f - \frac{D^2}{\gamma J} + D\bar{x} \tag{37}$$

$$\begin{aligned} &C_2x^2 + C_1x + C_0 = 0, \text{ where} \\ &C_2 = (v_0 - v_f - \frac{DT}{2})^2 \\ &C_1 = -2v_fT(DT + 2v_f) + T(v_0 + v_f)^2 + 4L(v_f - v_0) + 2LDT \\ &C_0 = \{2L - (v_0 + v_f)T\}\{2L - T(2v_f + DT)\} \end{aligned} \tag{38}$$

$$\frac{1}{4}Jx^2\gamma^2 + (v_0 - v_f - D\bar{x})\gamma + \frac{D^2}{J} = 0 \tag{39}$$

CASE ($T_4 = 0, T_2 > 0, T_6 = 0$)

$$v_p = v_0 - \frac{A^2}{\gamma J} + Ax = v_f + \frac{1}{4}\gamma J\bar{x}^2 \tag{40}$$

$$\begin{aligned} &C_2\bar{x}^2 + C_1\bar{x} + C_0 = 0, \text{ where} \\ &C_2 = (v_f - v_0 - \frac{AT}{2})^2 \\ &C_1 = -2v_0T(AT + 2v_0) + T(v_0 + v_f)^2 + 4L(v_0 - v_f) + 2LAT \\ &C_0 = \{2L - (v_0 + v_f)T\}\{2L - T(2v_0 + AT)\} \end{aligned} \tag{41}$$

$$\frac{1}{4}J\bar{x}^2\gamma^2 + (v_f - v_0 - Ax)\gamma + \frac{A^2}{J} = 0 \tag{42}$$

CASE ($T_4 = 0, T_2 = 0, T_6 = 0$)

$$v_p = v_0 + \frac{1}{4}\gamma Jx^2 = v_f + \frac{1}{4}\gamma J\bar{x}^2 \tag{43}$$

$$(v_0 - v_f)x^2 + \{(3v_f + v_0)T - 4L\}x + \{2L - (v_0 + v_f)T\}T = 0 \tag{44}$$

$$\gamma = \begin{cases} \dfrac{4(v_f - v_0)}{J(x^2 - \bar{x}^2)}, & v_f \ne v_0 \\ \dfrac{4(L/x - 2v_0)}{Jx^2}, & v_f = v_0 \end{cases} \tag{45}$$

Cases of constant velocity region

When the constant velocity region exists, v_p becomes V for AFP problem. Provided that the initial or final velocity is given by V, the solutions of x and γ are simply found. Therefore we subdivide the condition into 7 cases instead of 4 cases.

CASE ($T_4 > 0, T_2 > 0, T_6 > 0, v_0 \ne V, v_f \ne V$)

$$V = v_0 - \frac{A^2}{\gamma J} + Ax = v_f - \frac{D^2}{\gamma J} + D\bar{x} \tag{46}$$

$$\left(v_0 - V + \frac{D}{A}(v_f - V)\right)x + 2VT - 2L + \frac{1}{D}(v_f - V)\left(\frac{D^2}{A^2}(v_0 - V) - v_f + V\right) = 0 \tag{47}$$

$$\gamma = \frac{A^2}{J(Ax + v_0 - V)} \tag{48}$$

CASE ($T_4 > 0, T_2 = 0, T_6 > 0, v_0 \neq V, v_f \neq V$)

$$V = v_0 + \frac{1}{4}\gamma J x^2 = v_f - \frac{D^2}{\gamma J} + D\bar{x} \tag{49}$$

$$\frac{D(v_f - V)}{4(V - v_0)}x^2 + (v_0 - V)x + 2VT - 2L - \frac{1}{D}(v_f - V)^2 = 0 \tag{50}$$

$$\gamma = \frac{4(V - v_0)}{Jx^2} \tag{51}$$

CASE ($T_4 > 0, T_2 > 0, T_6 = 0, v_0 \neq V, v_f \neq V$)

$$V = v_0 - \frac{A^2}{\gamma J} + Ax = v_f + \frac{1}{4}\gamma J\bar{x}^2 \tag{52}$$

$$\frac{A(v_0 - V)}{4(V - v_f)}\bar{x}^2 + (v_f - V)\bar{x} + 2VT - 2L - \frac{1}{A}(v_0 - V)^2 = 0 \tag{53}$$

$$\gamma = \frac{4(V - v_f)}{J\bar{x}^2} \tag{54}$$

CASE ($T_4 > 0, T_2 = 0, T_6 = 0, v_0 \neq V, v_f \neq V$)

$$v = v_0 + \frac{1}{4}\gamma J x^2 = v_f + \frac{1}{4}\gamma J\bar{x}^2 \tag{55}$$

$$\left((v_0 - V)^2 - \frac{(v_f - V)^3}{(v_0 - V)}\right)x^2 - 4(L - VT)(v_0 - V)x + 4(L - VT)^2 = 0 \tag{56}$$

$$\gamma = \frac{4(V - v_0)}{Jx^2} \tag{57}$$

CASE ($T_4 > 0, v_0 = v_f = V$)
Since there are no acceleration and deceleration regions, the jerk scaling γ is meaningless, and we have the following.

$$x = \bar{x} = 0, \hat{x} = T \tag{58}$$

CASE ($T_4 > 0, v_0 = V, v_f \neq V$)
Since there is no acceleration region, $x = 0$ and $\bar{x}$ can be found from the displacement condition (6).

$$\bar{x} = \frac{2(L - VT)}{V - v_f} \tag{59}$$

We find the equations for γ by dividing the cases depending on whether or not constant acceleration region exists.

$$\gamma = \begin{cases} \dfrac{4(V - v_f)}{J\bar{x}^2} & , \dfrac{2(V - v_f)}{J\bar{x}} \leq \dfrac{D}{J} \\ \dfrac{D^2}{J(D\bar{x} - V + v_f)} & , \text{ otherwise} \end{cases} \tag{60}$$

CASE ($T_4 > 0, v_0 \neq V, v_f = V$)
With similar reasoning used in the previous case, we have the following conditions.

$$x = \frac{2(L - VT)}{V - v_0} \tag{61}$$

$$\gamma = \begin{cases} \dfrac{4(V - v_0)}{Jx^2} & , \dfrac{2(V - v_0)}{Jx} \leq \dfrac{A}{J} \\ \dfrac{A^2}{J(Ax - V + v_0)} & , \text{ otherwise} \end{cases} \tag{62}$$

Remark on the cases of no solutions
Even if the given traveling period T is longer than that of the time-optimal profile, a jerk limited solution may not exist because the displacement condition and the limit constraints for jerk and acceleration may contradict with one another. Actually, when both v_0 and v_f are positive and the time-optimal solution is AFP, or when both v_0 and v_f are negative and the time-optimal solution is DFP, time-fixed solutions do not exist for certain values of T. Such cases should be avoided.

5. CONCLUSIONS

Operation of robots for precision works at high speed requires not only high-performance feedback controllers but also smooth reference profile generation. Trajectories for industrial robots need continuous position, velocity, and acceleration profiles with their derivative limits imposed by the physical property of the actuators. Moreover, they often require real-time implementation.

In jerk limited trajectory generation, jerk limited velocity profile plays a very important role for both path motions and point-to-point motions. However, jerk limited velocity profile generation is an intricate problem, and no analytic solutions are found in the previous works.

In this paper we developed analytic solutions to the jerk limited profile generation problem by dividing the problems into several cases using the displacement conditions and the peak velocity conditions. Since no iterative methods are used for solutions, the proposed solutions are essentially ready for real-time implementation.

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