Forward transformation

The transformation can be defined by the sequential multiplication of the matrices:

$$ {}^{w}A_{t} = {}^{w}A_{O} \cdot {}^{O}A_{A} \cdot {}^{A}A_{P} \cdot {}^{P}A_{t} \tag{18} $$

with the matrices built up as follows:

$$ {}^{w}A_{O} = \begin{bmatrix} C_B & 0 & S_B & 0 \\ 0 & 1 & 0 & 0 \\ -S_B & 0 & C_B & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{O}A_{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & D_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{19} $$

$$ {}^{A}A_{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_A & -S_A & 0 \\ 0 & S_A & C_A & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{p}A_{T} = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y - D_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{20} $$

Executing the matrix multiplications starting from the left:

$$ {}^{w}A_{A} = {}^{w}A_{O} \cdot {}^{O}A_{A} $$

$$ {}^{w}A_{A} = \begin{bmatrix} C_B & 0 & S_B & 0 \\ 0 & 1 & 0 & 0 \\ -S_B & 0 & C_B & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & D_y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_B & 0 & S_B & 0 \\ 0 & 1 & 0 & D_y \\ -S_B & 0 & C_B & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{w}A_{P} = {}^{w}A_{A} \cdot {}^{A}A_{P} $$

$$ {}^{w}A_{P} = \begin{bmatrix} C_B & 0 & S_B & 0 \\ 0 & 1 & 0 & D_y \\ -S_B & 0 & C_B & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_A & -S_A & 0 \\ 0 & S_A & C_A & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_B & S_A S_B & S_B C_A & 0 \\ 0 & C_A & -S_A & D_y \\ -S_B & S_A C_B & C_A C_B & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$^{w}A_t = {}^{w}A_P \cdot {}^{P}A_t$$

$$^{w}A_t = \begin{bmatrix} C_B & S_A S_B & S_B C_A & 0 \\ 0 & C_A & -S_A & D_y \\ -S_B & S_A C_B & C_A C_B & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y - D_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{w}A_t = \begin{bmatrix} C_B & S_A S_B & S_B C_A & C_B P_x + S_A S_B (P_y - D_y) + S_B C_A P_z \\ 0 & C_A & -S_A & C_A (P_y - D_y) - S_A P_z + D_Y \\ -S_B & S_A C_B & C_A C_B & -S_B P_x + S_A C_B (P_y - D_y) + C_A C_B P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (21)$$

$$Q = \begin{bmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{bmatrix} = \begin{bmatrix} C_B P_x + S_A S_B (P_y - D_y) + S_B C_A P_z \\ C_A (P_y - D_y) - S_A P_z + D_Y \\ -S_B P_x + S_A C_B (P_y - D_y) + C_A C_B P_z \\ 1 \end{bmatrix} \quad (25)$$

$$Q = \begin{bmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{bmatrix} = \begin{bmatrix} C_B & S_A S_B & S_B C_A & -S_A S_B D_y \\ 0 & C_A & -S_A & -C_A D_y + D_y \\ -S_B & S_A C_B & C_A C_B & -S_A C_B D_y \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} = {}^{Q}A_P \cdot P \quad (26)$$

Inverse transformation

Using $$^{q}A_p=\begin{bmatrix} R & q \\ 0 & 1 \end{bmatrix}$$

and (26)

We extract the rotation matrix *R* and the translation vector *q:*

$$R=\begin{bmatrix} C_B & S_A S_B & S_B C_A \\ 0 & C_A & -S_A \\ -S_B & S_A C_B & C_A C_B \end{bmatrix} \qquad q=\begin{bmatrix} -S_A S_B D_y \\ -C_A D_y + D_y \\ -S_A C_B D_y \end{bmatrix}$$

Using $$\left(^{q}A_p\right)^{-1}=\begin{bmatrix} R^T & -R^T q \\ 0 & 1 \end{bmatrix}$$

$$-R^T=\begin{bmatrix} -C_B & 0 & S_B \\ -S_A S_B & -C_A & -S_A C_B \\ -S_B C_A & S_A & -C_A C_B \end{bmatrix}$$

$$-R^T q=\begin{bmatrix} -C_B & 0 & S_B \\ -S_A S_B & -C_A & -S_A C_B \\ -S_B C_A & S_A & -C_A C_B \end{bmatrix}\cdot\begin{bmatrix} -S_A S_B D_y \\ -C_A D_y + D_y \\ -S_A C_B D_y \end{bmatrix}=\begin{bmatrix} 0 \\ S_A^2 S_B^2 D_y + C_A^2 D_y - C_A D_y + S_A^2 C_B^2 D_y \\ S_A S_B^2 C_A D_y - S_A C_A D_y - S_A D_y + S_A C_A C_B^2 D_y \end{bmatrix}$$

Using $\sin^2(\theta) + \cos^2(\theta) = 1$

$$-R^T q=\begin{bmatrix} 0 \\ S_A^2\left(S_B^2 D_y + C_B^2 D_y\right) + C_A^2 D_y + C_A D_y \\ S_A C_A D_y\left(S_B^2 + C_B^2\right) - S_A C_A D_y + S_A D_y \end{bmatrix}=\begin{bmatrix} 0 \\ S_A^2\left(S_B^2 D_y + C_B^2 D_y\right) + C_A^2 D_y + C_A D_y \\ S_A C_A D_y - S_A C_A D_y + S_A D_y \end{bmatrix}$$

$$-R^T q=\begin{bmatrix} 0 \\ S_A^2\left(S_B^2 D_y + C_B^2 D_y\right) + C_A^2 D_y - C_A D_y \\ S_A D_y \end{bmatrix}=\begin{bmatrix} 0 \\ \left(S_A^2 + C_A^2\right) D_y - C_A D_y \\ S_A D_y \end{bmatrix}$$

$$-R^T q=\begin{bmatrix} 0 \\ D_y - C_A D_y \\ S_A D_y \end{bmatrix}$$

$$P = \left({}^{Q}A_P\right)^{-1} \cdot Q = \begin{bmatrix} R^T & -R^T q \\ 0 & 1 \end{bmatrix} \cdot Q$$

$$P = \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} = \begin{bmatrix} C_B & 0 & -S_B & 0 \\ S_A S_B & C_A & S_A C_B & D_y - C_A D_y \\ S_B C_A & -S_A & C_A C_B & S_A D_y \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{bmatrix} \quad (27)$$

$$\begin{aligned} P_x &= C_B Q_x - S_B Q_z \\ P_y &= S_A S_B Q_x + C_A Q_y + S_A C_B Q_z - C_A D_y + D_y \\ P_z &= S_B C_B Q_x - S_A Q_y + C_A C_B Q_z + S_A D_y \end{aligned} \quad (28)$$